

$$\begin{aligned}
 B &= A^{-1}C \\
 &= \begin{bmatrix} -5 & 2 & 1 \\ 2 & -1 & 0 \\ 4 & -1 & -1 \end{bmatrix} \begin{bmatrix} 46 & 28 & 4 & 30 & 29 & 38 \\ 84 & 47 & 5 & 48 & 57 & 57 \\ 85 & 46 & 10 & 72 & 38 & 95 \end{bmatrix} \\
 &= \begin{bmatrix} 23 & 0 & 0 & 18 & 7 & 19 \\ 8 & 9 & 3 & 12 & 1 & 19 \\ 15 & 19 & 1 & 0 & 21 & 0 \end{bmatrix}
 \end{aligned}$$

Writing the numbers in the columns of this matrix in sequence and using the correspondence between numbers and letters noted earlier produces the decoded message:

23 8 15 0 9 19 0 3 1 18 12 0 7 1 21 19 19 0
 W H O I S C A R L G A U S S

The answer to this question can be found earlier in this chapter.

Matched Problem 5 The message below was also encoded with the matrix A in Example 5. Decode this message:

46 84 85 28 47 46 32 41 78 25 42 53 25 37 63 43 71 83 19 37 25

Exercises 4.5

Skills Warm-up Exercises

W In Problems 1–4, find the additive inverse and the multiplicative inverse, if defined, of each real number. (If necessary, review Section A.1).

- (A) 4 (B) -3 (C) 0
- (A) -7 (B) 2 (C) -1
- (A) $\frac{2}{3}$ (B) $-\frac{1}{7}$ (C) 1.6
- (A) $\frac{4}{5}$ (B) $\frac{12}{7}$ (C) -2.5

In Problems 5–8, does the given matrix have a multiplicative inverse? Explain your answer.

- $\begin{bmatrix} 2 & 5 \end{bmatrix}$
- $\begin{bmatrix} 4 \\ 8 \end{bmatrix}$
- $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
- $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

In Problems 9–18, find the matrix products. Note that each product can be found mentally, without the use of a calculator or pencil-and-paper calculations.

- (A) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix}$ (B) $\begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
- (A) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 6 \\ 5 & 2 \end{bmatrix}$ (B) $\begin{bmatrix} -1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

- (A) $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix}$ (B) $\begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$
- (A) $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 6 \\ 5 & 2 \end{bmatrix}$ (B) $\begin{bmatrix} -1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$
- (A) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix}$ (B) $\begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- (A) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 6 \\ 5 & 2 \end{bmatrix}$ (B) $\begin{bmatrix} -1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 & 3 \\ 2 & 4 & -2 \\ 5 & 1 & 0 \end{bmatrix}$
- $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -4 & 0 \\ 1 & 2 & -5 \\ 6 & -3 & -1 \end{bmatrix}$
- $\begin{bmatrix} -2 & 1 & 3 \\ 2 & 4 & -2 \\ 5 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- $\begin{bmatrix} 3 & -4 & 0 \\ 1 & 2 & -5 \\ 6 & -3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

In Problems 19–28, examine the product of the two matrices to determine if each is the inverse of the other.

- $\begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}; \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$
- $\begin{bmatrix} -2 & -1 \\ -4 & 2 \end{bmatrix}; \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}$

21. $\begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}; \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$
22. $\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}; \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$
23. $\begin{bmatrix} -5 & 2 \\ -8 & 3 \end{bmatrix}; \begin{bmatrix} 3 & -2 \\ 8 & -5 \end{bmatrix}$
24. $\begin{bmatrix} 7 & 4 \\ -5 & -3 \end{bmatrix}; \begin{bmatrix} 3 & 4 \\ -5 & -7 \end{bmatrix}$
25. $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}; \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$
26. $\begin{bmatrix} 1 & 0 & 1 \\ -3 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}; \begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$
27. $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{bmatrix}; \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$
28. $\begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}; \begin{bmatrix} 1 & 0 & -1 \\ -3 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

 Without performing any row operations, explain why each of the matrices in Problems 29–38 does not have an inverse.

29. $\begin{bmatrix} 1 & 2 & 0 \\ -3 & 2 & -1 \end{bmatrix}$
30. $\begin{bmatrix} -2 & 3 & -1 \\ 4 & 0 & 1 \end{bmatrix}$
31. $\begin{bmatrix} 1 & -2 \\ 3 & 0 \\ 2 & -1 \end{bmatrix}$
32. $\begin{bmatrix} 0 & -1 \\ 2 & -2 \\ 1 & -3 \end{bmatrix}$
33. $\begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}$
34. $\begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$
35. $\begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix}$
36. $\begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$
37. $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$
38. $\begin{bmatrix} -2 & -3 \\ 4 & 6 \end{bmatrix}$

Given M in Problems 39–48, find M^{-1} and show that $M^{-1}M = I$.

39. $\begin{bmatrix} -1 & 0 \\ -3 & 1 \end{bmatrix}$
40. $\begin{bmatrix} 1 & -5 \\ 0 & -1 \end{bmatrix}$
41. $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$
42. $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$
43. $\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$
44. $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$
45. $\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 1 \\ 2 & -1 & 4 \end{bmatrix}$
46. $\begin{bmatrix} 2 & 3 & 0 \\ 1 & 2 & 3 \\ 0 & -1 & -5 \end{bmatrix}$
47. $\begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & -1 \\ 1 & 0 & 2 \end{bmatrix}$
48. $\begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & 0 \\ 1 & 1 & -2 \end{bmatrix}$

Find the inverse of each matrix in Problems 49–54, if it exists.

49. $\begin{bmatrix} 4 & 3 \\ -3 & -2 \end{bmatrix}$
50. $\begin{bmatrix} -4 & 3 \\ -5 & 4 \end{bmatrix}$
51. $\begin{bmatrix} 2 & 6 \\ 3 & 9 \end{bmatrix}$
52. $\begin{bmatrix} 2 & -4 \\ -3 & 6 \end{bmatrix}$
53. $\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$
54. $\begin{bmatrix} -5 & 3 \\ 2 & -2 \end{bmatrix}$

In Problems 55–60, find the inverse. Note that each inverse can be found mentally, without the use of a calculator or pencil-and-paper calculations.

55. $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$
56. $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$
57. $\begin{bmatrix} 3 & 0 \\ 0 & -5 \end{bmatrix}$
58. $\begin{bmatrix} -2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$
59. $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -8 \end{bmatrix}$
60. $\begin{bmatrix} 3 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -5 \end{bmatrix}$

Find the inverse of each matrix in Problems 61–68, if it exists.

61. $\begin{bmatrix} -5 & -2 & -2 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$
62. $\begin{bmatrix} 2 & -2 & 4 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$
63. $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix}$
64. $\begin{bmatrix} 1 & -1 & 0 \\ 2 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$
65. $\begin{bmatrix} -1 & -2 & 2 \\ 4 & 3 & 0 \\ 4 & 0 & 4 \end{bmatrix}$
66. $\begin{bmatrix} 4 & 2 & 2 \\ 4 & 2 & 0 \\ 5 & 0 & 5 \end{bmatrix}$
67. $\begin{bmatrix} 2 & -1 & -2 \\ -4 & 2 & 8 \\ 6 & -2 & -1 \end{bmatrix}$
68. $\begin{bmatrix} -1 & -1 & 4 \\ 3 & 3 & -22 \\ -2 & -1 & 19 \end{bmatrix}$
69. Show that $(A^{-1})^{-1} = A$ for: $A = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$

70. Show that $(AB)^{-1} = B^{-1}A^{-1}$ for

$$A = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}$$

 71. Discuss the existence of M^{-1} for 2×2 diagonal matrices of the form

$$M = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$$

Generalize your conclusions to $n \times n$ diagonal matrices.

 72. Discuss the existence of M^{-1} for 2×2 upper triangular matrices of the form

$$M = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$$

Generalize your conclusions to $n \times n$ upper triangular matrices.